LINEAR PROGRAMMING

Introduction

Linear programming comes under the allocation problem, is a problem which involves the allocation of given number of resources to the job. The objective of these problems is to optimize the total effectiveness i.e to minimize the total cost or maximize the total return. Generally, there are three types of allocation problem;

i. Linear programming problem

ii. Transportation problem

iii. Assignment problem

Linear Programming Problem

Before formally defining a linear programming problem, we define the concepts of linear function and linear inequality. A function $f(x_1, x_2, ..., x_n)$ of $x_1, x_2, ..., x_n$ is a linear function if and only if for some set of constants $c_1, c_2, ..., c_n, f(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 +, ... + c_nx_n$. For any linear function $f(x_1, x_2, ..., x_n)$ and any number b, the inequalities $f(x_1, x_2, ..., x_n) \leq b$ and $f(x_1, x_2, ..., x_n) \geq b$ are linear inequalities.

Thus, $2x_1 + 3x_2 \leq 3$ and $2x_1 + x_2 \geq 3$ are linear inequalities, but $x^2x_2 \geq 3$ is not a linear inequality.

Now, the term Linear Programming is the combination of the two term 'Linear' and 'Programming'. The term linear means that all the relations in the particular problem are linear and the term programming refers to the process determining particular programme or plan of action.

Therefore, linear programming method is a technique of choosing the best alter- native from the set of feasible alternatives, in the situations in which the objective functions as well as constraints can be expressed as linear mathematical function.

The linear function which is to be optimized is called the objective function and the conditions of the problem expressed as simultaneous linear equations (or in- equalities) are referred as constraints.

Thus, generally we define Linear Programming as the process of transforming a real life problem into a mathematical model which contains variables representing decisions that can be examined and solved for an optimal solution using algorithms.

Mathematical Formulation of a LPP

A general linear programming problem can be stated as follows; Find $x_1, x_2, x_3, \ldots, x_n$ which optimize the linear function.

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subjected to the constraints

$$\begin{array}{lllll}
a_{11}x_1 & +a_{12}x_2 & +\ldots & +a_{1j}x_j & +\ldots a_{1n}x_n(\leq = \geq)b_1 \\
a_{21}x_1 & +a_{22}x_2 & +\ldots & +a_{2j}x_j & +\ldots a_{2n}x_n(\leq = \geq)b_2 \\
. & . & . & . & . \\
a_{i1}x_1 & +a_{i2}x_2 & +\ldots & +a_{ij}x_j & +\ldots a_{in}x_n(\leq = \geq)b_i \\
. & . & . & . & . \\
a_{m1}x_1 & +a_{m2}x_2 & +\ldots & +a_{mi}x_j & +\ldots a_{mn}x_n(\leq = \geq)b_n
\end{array}$$

and non negativity constraints

$$x_j \geq 0, 1, 2, 3, \ldots, n$$

Where all a_{ij}, b_i and C_j are constants and x_i are variables.

Matrix Form of LPP

The LPP can be expressed in the form of matrix as follows;
Maximize or Minimize $Z = CX$ is the objective function.
Subject to
$AX(\leq = \geq)b$ constraints equation, $b > 0, X \geq 0$ non negativity restrictions. Where $X = (x_1, x_2, \ldots, x_n)$ and $C = (c_1, c_2, \ldots, c_n)$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ . & . & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

Procedure for Formulation of LP Problems

1. To write down the decision variables of the problem.
2. To formulate the objective function to be optimized (Maximized or Minimized) as a linear function of the decision variable.
3. To formulate the other conditions of the problem such as resource limitation, market constraints, interrelations between variables etc., as linear equations in terms of decision variables.
4. To add non negativity constraints from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraints together form a linear programming problem.

Examples 1
A resourceful home decorator manufactures two types of lamps says A and B. Both lamps go through two technician's first a cutter, second a finisher. Lamp A requires 2 hrs of the cutter's time and 1 hr of the finisher's time. The cutter has 104 hrs and finisher has 76 hrs of available time each month. Profit per lamp A is Rs.600 and per B lamp is Rs.1100. Assuming that he can sale all that he produces, how many of each type of lamps should be manufactured to obtain the best return.

Solution: Formulation of the Mathematical Model of the Problem.
For the clear understanding of the problem, first construct a table;

Lamps	Cutter	Finisher	Profit
A	2 hrs	1 hr	Rs.600
B	1 hr	2 hrs	Rs.1100
Available time	104 hrs	76 hrs	

Decision Variables:
Let the decorator manufacture x_1 and x_2 lamps of type A and B respectively.
Objective Functions:
Therefore, the total profit (in Rs.) has to be maximized.

$$Max(z) = 6x_1 + 11x_2$$

Constraints
The manufacturer has limited time for manufacturing the lamp. There are 104

hrs available for cutting and 76 hrs available for finishing. Thus, total processing time is restricted.

$$2x_1 + x_2 \leq 104$$
$$x_1 + x_2 \leq 76$$

Finally the complete LPP is;

$$Max(z) = 6x_1 + 11x_2$$

Subject to

$$2x_1 + x_2 \leq 104$$
$$x_1 + x_2 \leq 76$$

and

$$x_1, x_2 \geq 0$$

Example 2: Abdallah, a retired government officer, has recently received his retirement benefits, viz., provident fund, gratuity, etc. He is contemplating how much money he should invest in various alternatives open to him so as to maximize return on investment. The investment alternatives are Government securities, fixed deposits of a public limited company, equity shares, time deposits in a bank, and house construction. He has made a subjective estimate of the risk involved on a five-point scale. The data on the return on investment, the number of years for which the funds will be blocked to earn this return on investment and the subjective risk involved are as follows;

	Return(%)	No. of Years	Risk
Government securities	6	15	1
Company deposits	13	3	3
Time deposits	10	5	2
Equity shares	20	6	5
House construction	25	10	1

He was wondering as to what percentage of funds he should invest in each alternative so as to maximize the return on investment. He decided that the risk should not be more than 4, and funds should not be locked up for more than 15 years. He would necessarily invest at least 25% in house construction. Formulate this problem as an LP model.

Solution: Formulation of an LP model

Decision Variables:

Let x_1, x_2, x_3, x_4 and x_5 be percentage of the total fund that should be invested in all given five schemes, respectively.

Objective Functions:

Therefore, the objective function is to maximize the return on investment.

$$Max\ Z = 6x_1 + 13x_2 + 10x_3 + 20x_4 + 25x_5$$

Constraints:

$$\begin{aligned} 15x_1 + 3x_2 + 5x_3 + 6x_4 + 10x_5 &\leq 15 \\ x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 &\leq 4 \\ x_5 &\geq 0.25 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \end{aligned}$$

Finally the complete LP Model is;

$$Max\ Z = 6x_1 + 13x_2 + 10x_3 + 20x_4 + 25x_5$$

Subject to

$$\begin{aligned} 15x_1 + 3x_2 + 5x_3 + 6x_4 + 10x_5 &\leq 15 \\ x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 &\leq 4 \\ x_5 &\geq 0.25 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \end{aligned}$$

and

$$x_j \geq 0 \quad for\ \ all\ j$$

Some Important Definitions in LPP

Consider the following LPP

$$Optimize\ z = CX$$

subject to

$$AX(\leq = \geq)b$$

and

$$X \geq 0$$

(i) Objective Function;

Is the function

$$z = CX = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

which is to be optimized (maximized or minimized).

(ii) Decision Variables;

The variables $x_1, x_2, \ldots, x_n$ whose values are to be determined are called decision variables.

(iii) Cost (profit) Coefficients;

The coefficients $c_1, c_2, \ldots, c_n$ are called cost (profit) coefficients.

(iv) Requirements;

The constraints $b_1, b_2, \ldots, b_n$ are called requirements.

(v) Solution;

A set of real values of $X = (x_1, x_2, \ldots, x_n)$ which satisfies the constraint $AX(\leq = \geq)b$ is called solution.

(vi) Feasible Solution

A set of real values of $X = (x_1, x_2, \ldots, x_n)$ which

Satisfies the constraints $AX(\leq = \geq)b$ and

Satisfies the non-negativity restriction $X \geq 0$ is called feasible solution.

(vii) Optimal Solution;

A set of real values of $X = (x_1, x_2, \ldots, x_n)$ which

Satisfies the constraints $AX(\leq = \geq)b$

Satisfies the non-negativity restriction $X \geq 0$ and

Optimizes the objective function $Z = CX$ is called optimal solution.

(viii) Results;

If an LPP has many optimal solutions, it is said to have multiple solutions.

If an LPP has only one optimal solution, it is said to have unique solution.

There may be a case where the LPP may not have any feasible solution at all (no solution).

For some LPP the optimum value of Z may be infinity. In this case the LPP is said to have unbounded solution.

Generally

Objective function is a Mathematical expression that describes the project objectives.

Constraints are mathematical expressions that describe constraints eg. capacity constraints.